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LARGE DEFLECTIONS OF CIRCULAR PLATES OF VARIABLE THICKNESS

B. BANERJEE
Head Department of Mathematics, Jalpaiguri Govt. Engineering College, Jalpaiguri, West Bengal, India

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Abstract—Large deflections of a clamped circular plate of variable thickness have been investigated following the equations of Banerjee and Dutta. Numerical results obtained have been compared with other known results.

INTRODUCTION

Thin plates of different shapes frequently occur in many structures and study of bending properties of a plate is imperative to a design engineer. With the increased use of strong and light weight structures, especially in aerospace engineering and in the study of vibrations of machine parts, many problems of non-linear deformations naturally arise where the supplementary stresses in the middle plane of the plate must be taken into account in deriving the differential equations of plates.

The coupled non-linear partial differential equations for large amplitude axisymmetric deformations were initially derived by von-Karman[1]. The von-Karman equations in the coupled form are difficult to solve and different numerical methods have been followed by Schmidt[2], Nash and Cooley[3], Nowinski[4] and many other authors for investigation of large deflections of plates.

An approximate method for solving the large deflections of plates has been proposed by Berger[5]. This method is based on the neglect of e_2, the second invariant of the middle surface strains, in the expression corresponding to the total potential energy of the system. An advantage of Berger's method is that the coupled differential equations are decoupled if e_2 is neglected. Nowinski[6], Nash and Modeer[7], Banerjee[8, 9], Sinha[10], Datta[11, 12] and many other authors followed Berger's method for solution of various large deflection plate problems involving static, dynamic as well as thermal loadings with ease and sufficient accuracy. Nowinski and Ohnabe[13] pointed out certain inaccuracies in Berger's equations and concluded that Berger's line of thought leads to meaningless results for movable edge conditions. This is due to the fact that the neglect of e_2 for movable edges fails to imply freedom of rotation in the meridian planes where the membrane stress

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right]$$

exists. For movable edges the in-plane displacement u is never zero and thus Berger's equations lead to absurd results. On the other hand, for immovable clamped edge, $u = 0$ and $dw/dr = 0$ at the boundary and therefore, Berger's equations give sufficiently accurate results. For simply supported immovable edges, $u = 0$ but $dw/dr \neq 0$. Thus Berger's equations give fairly accurate results.

It is also interesting to note that under many loading conditions, especially uniform and under relatively "smooth" and regular boundary conditions—the distortional energy and its variation should be substantially smaller than the dilatiational. Hence the "Berger assumption" which too simplistically has been translated into assuming a Poisson's ratio of unity and hence patently absurd. For this reason this assumption has always yielded reasonably good practical results for the uniform, or smoothly varying loading. The circular plate is the best geometry, but as any in-plane large distortional changes even in rectangular plates is usually confined to the corners, reasonable results should also be expected there. On the other hand disparities such as a movable boundary suggest large energy changes and the basic hypothesis becomes questionable.

Banerjee and Datta[14] suggested a modified energy expression by bringing directly the expression for σ_{rr} in the total potential energy of the system. A new set of differential equations has been obtained in a decoupled form. The accuracy of these equations has been tested for a circular and a square plate under different boundary conditions and satisfactory results have been obtained.

The equivalent hypothesis of the study[14] is that the radial stretching of the plate is proportional to $(dw/dr)^2$. This is certainly reasonable because under any type of loading and under any boundary condition the extra strain imposed by bending is represented by $(dw/dr)^2$. Also this hypothesis has effectively linearised the problem by connecting the in-plane and bending deflection. In fact, any hypothesis which connects the in-plane and bending deflection should effectively linearise the problem. For example, in a somewhat similar case, if the compressibility is assured, i.e. $\nu = 1/2$ (for rubber) then the first strain invariant must vanish—thus prescribing a unique relation between u and w. It is further a point of interest that the variational calculation and definition of the constant A has the physical nature of the in-plane stress and this vanishes for the movable boundary.

Plates of non-uniform thickness are sometimes encountered in the design of machine parts, such as diaphragms of steam turbines and pistons of reciprocating engines. Investigations of plates of non-uniform thickness based on linear theory have been done by many workers and the bibliography of these workers are given in [15]. As far as it is known only one paper by Banerjee[16] can be located where large deflection of a clamped circular plate of variable thickness has been investigated using von-Karman's equations.

In this paper the large deflection of a clamped circular plate of variable thickness has been discussed following the line of thought as given in[14]. A new set of differential equations has been formed in a decoupled form considering the plate thickness varying exponentially. The results obtained have been given in tabular form and compared with other known results.

FORMULATION OF EQUATION

In polar co-ordinates, the total potential energy, V of a thin isotropic circular plate of radius a, and of thickness h is given by

$$V = \frac{1}{2}\int_0^a D\left[\left(\frac{d^2w}{dr^2}\right)^2 + \frac{2\nu}{r}\frac{dw}{dr}\frac{d^2w}{dr^2} + \frac{1}{r^2}\left(\frac{dw}{dr}\right)^2 + \frac{12}{h^2}\{e_1{}^2 + 2(\nu-1)e_2\}\right] r\,dr - \int_0^a qwr\,dr \quad (1)$$

where D is the flexural rigidity of the plate given by $D = Eh^3/12(1-\nu^2)$, w is the deflection, ν Poisson's ratio, and e_1 and e_2 the first and second invariant of the middle surface strains respectively given by

$$e_1 = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{u}{r}$$

$$e_2 = \frac{u}{r}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right].$$

Here, u is the in-plane displacement and q is the uniform static load. Equation (1) may be rewritten in the following form

$$V = \frac{1}{2}\int_0^a D\left[\left(\frac{d^2w}{dr^2}\right)^2 + \frac{2\nu}{r}\frac{dw}{dr}\frac{d^2w}{dr^2} + \left(\frac{1}{r}\frac{dw}{dr}\right)^2 + \frac{12}{h^2}\left\{\bar{e}_1{}^{-2} + (1-\nu^2)\frac{u^2}{r^2}\right\}\right] r\,dr - \int_0^a qwr\,dr \quad (2)$$

where

$$\bar{e}_1 = \frac{du}{dr} + \nu\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2.$$

If the term $(1-\nu^2)(u^2/r^2)$ in (2) is replaced by $(\lambda/4)(dw/dr)^4$ λ being a factor depending on the

Poisson's ratio for the plate material, decoupling of (2) is possible. Introducing the term $\lambda/4(\mathrm{d}w/\mathrm{d}r)^4$ in place of $(1-\nu^2)u^2/r^2$, remembering the plate thickness h as a variable quantity and applying Euler's variational method to (2) one get the following differential equation,

$$\frac{\mathrm{d}^4w}{\mathrm{d}r^4}+2\frac{\mathrm{d}^3w}{\mathrm{d}r^3}\left[\frac{1}{r}+\frac{3}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\right]+\frac{\mathrm{d}^2w}{\mathrm{d}r^2}\left[\frac{1}{r^2}+\frac{3(\nu+2)}{hr}\frac{\mathrm{d}h}{\mathrm{d}r}+\frac{6}{h^2}\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2+\frac{3}{h}\frac{\mathrm{d}^2h}{\mathrm{d}r^2}\right]$$
$$+\frac{\mathrm{d}w}{\mathrm{d}r}\left[\frac{1}{r^3}-\frac{3\nu}{hr^2}\frac{\mathrm{d}h}{\mathrm{d}r}+\frac{6\nu}{h^2r}\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2+\frac{3\nu}{hr}\frac{\mathrm{d}^2h}{\mathrm{d}r^2}\right]$$
$$-\frac{12}{h^3}Ar^{\nu-1}\left[\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\frac{\nu}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right]-\frac{6\lambda}{h^3}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\left[3h\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\frac{\mathrm{d}h}{\mathrm{d}r}\frac{\mathrm{d}w}{\mathrm{d}r}+\frac{h}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right]=\frac{12(1-\nu^2)}{Eh^3}q \tag{3}$$

where A is determined from

$$h\left[\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{\nu u}{r}+\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right]=Ar^{\nu-1}. \tag{4}$$

For movable edge, $A=0$.

λ has been chosen approximately from the condition $\partial v/\partial\lambda=0$ for minimum potential energy v. For clamped edge it has been assumed that $\mathrm{d}u/\mathrm{d}r\approx 1/2[1/2(\mathrm{d}w/\mathrm{d}r)^2]$ and for simple-support $\mathrm{d}u/\mathrm{d}r\approx 0$. For clamped edge $\lambda=2\nu^2$. For simple support $\lambda=\nu^2$, $\nu=$ Poisson's ratio. Let

$$W=W_0\left[1-\frac{r^2}{a^2}\right]^2 \tag{5}$$

for clamped edge at $r=a$. Let

$$h=h_0e^{\frac{-\beta r^2}{6a^2}} \tag{6}$$

be the thickness variation. This type of variation is useful in design and has been discussed fully in [15] for corresponding small deflection problem.

Putting (5) and (6) in (4) and integrating we get

$$A=\frac{8w_0^2a^{-\nu-1}h_0\left[\frac{1}{7+\nu}+\frac{1}{3+\nu}-\frac{2}{5+\nu}\right]}{\sum_{m=0}^{\infty}\left(\frac{\beta}{6}\right)^m\frac{1}{\underline{|m}(2m+2\nu)}}. \tag{7}$$

Putting (6) and (5) in (3) and applying Galerkin's procedure one gets the following cubic equation determining W_0.

$$\frac{W_0}{h_0}\left[\frac{32}{3}-\frac{3}{2}\beta+\frac{\beta^2}{30}+\frac{\nu\beta}{6}-\frac{\nu\beta^2}{10}\right]-\frac{W_0^3}{h_0^3}\frac{384\left[\frac{1}{7+\nu}+\frac{1}{3+\nu}-\frac{2}{5+\nu}\right]}{\sum_{m=0}^{\infty}\left(\frac{\beta}{6}\right)^m\frac{1}{\underline{|m}(2m+2\nu)}}\left[\sum_{s=0}^{\infty}\left(\frac{\beta}{2}\right)^s\frac{1}{\underline{|s}}\left\{-\frac{\nu+1}{2s+\nu+1}\right.\right.$$
$$\left.\left.+\frac{5+3\nu}{2s+\nu+3}-\frac{7+3\nu}{2s+\nu+5}+\frac{3+\nu}{2s+\nu+7}\right\}\right]$$
$$-\frac{6\lambda W_0^3}{h_0^3}\left[\sum_{s=0}^{\infty}\left(\frac{\beta}{2}\right)^s\frac{1}{s}\left\{-\frac{1}{s+2}+\frac{13}{2s+6}-\frac{16}{s+4}+\frac{19}{s+5}-\frac{11}{s+6}+\frac{5}{2s+14}+\frac{\beta}{6(2s+6)}-\frac{5\beta}{(2s+8)6}\right.\right.$$
$$\left.\left.+\frac{5\beta}{3(2s+10)}-\frac{5\beta}{3(2s+12)}+\frac{5\beta}{6(2s+14)}-\frac{\beta}{6(2s+16)}\right\}\right]$$
$$=\frac{12(1-\nu^2)}{Eh_0^4}qa^4\left[\sum_{s=0}^{\infty}\left(\frac{\beta}{2}\right)^s\frac{1}{s}\left\{\frac{1}{2s+2}-\frac{1}{\underline{s+2}}+\frac{1}{2s+6}\right\}\right]. \tag{8}$$

Table 1.

Boundary condition	A_1 Immovable edge Present study	Ref. [16]	B_1 Present study and Ref. [16]	β	A_1 Movable edge Present study	Ref. [16]	B_1 Present study and Ref. [16]
Plate	0.46	0.471	0.171	0	0.12	0.146	0.171
Clamped	0.57	0.597	0.217	1	0.18	0.203	0.217
	0.70	0.728	0.275	2	0.23	0.256	0.275
	0.795	0.824	0.349	3	0.28	0.306	0.349

$\nu = 0.3$

The above equation is of the following form

$$\frac{W_0}{h_0} + A_1\frac{W_0^3}{h_0^3} = B_1\frac{qa^4}{Eh_0^4}. \tag{9}$$

NUMERICAL RESULTS AND DISCUSSIONS

Numerical values of the co-efficients A_1 and B_1 in eqn (9) have been calculated for different values of β and are presented in the following Table for comparison with other known results.

It is observed that the deflections are always higher than those obtained theoretically. It is clear from the Table that the results of the present study are in very good agreement with those obtained in [16] where von-Karman's equations have been employed and the method of solution is laborious. The following are the advantages of the present study.

(1) Unlike von-Karman's equations, the equations of the present study are decoupled and hence they can be solved without difficulty.

(2) Unlike Berger's equations, the equations of the present study are valid both for movable and immovable edge conditions.

(3) The results can be obtained with ease and accuracy and without much computational labour.

(4) From the same cubic equation determining w_0, the results of both movable and immovable edge conditions can be obtained.

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